

# Trig addition formulas.

Euler's formula:

$$\boxed{e^{i\theta} := \cos\theta + i\sin\theta} \quad (i^2 = -1)$$

Law of exponents for e raised to imaginary arguments:

$$\boxed{[e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)}]}, \quad (*)$$

that is

$$\begin{aligned} & (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \\ &= \cos\theta \cos\phi - \sin\theta \sin\phi + \\ &+ i(\cos\theta \sin\phi + \sin\theta \cos\phi) \\ &= \cos(\theta + \phi) + i\sin(\theta + \phi), \end{aligned}$$

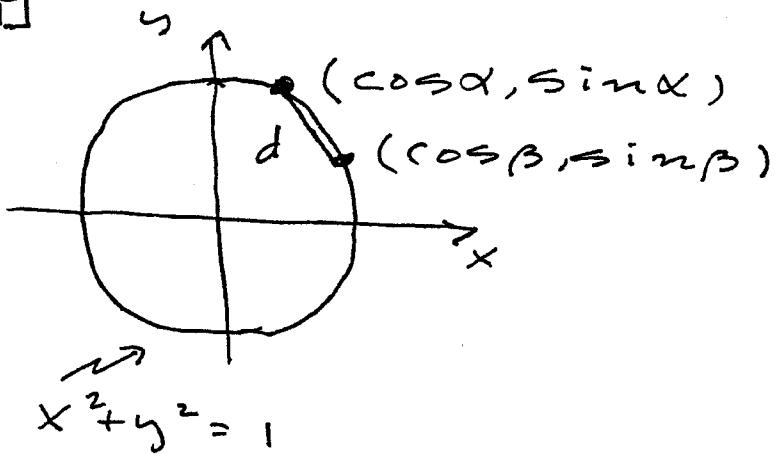
that is

$$\boxed{\begin{aligned} \cos(\theta + \phi) &= \cos\theta \cos\phi - \sin\theta \sin\phi \\ \sin(\theta + \phi) &= \cos\theta \sin\phi + \sin\theta \cos\phi \end{aligned}}$$

We'll prove Euler's formula in MATH14 via power series expansion of  $e^z$ ,  $z$  complex. For now just accept Euler's formula as the definition of  $e^{i\theta}$  ( $\theta$  real).

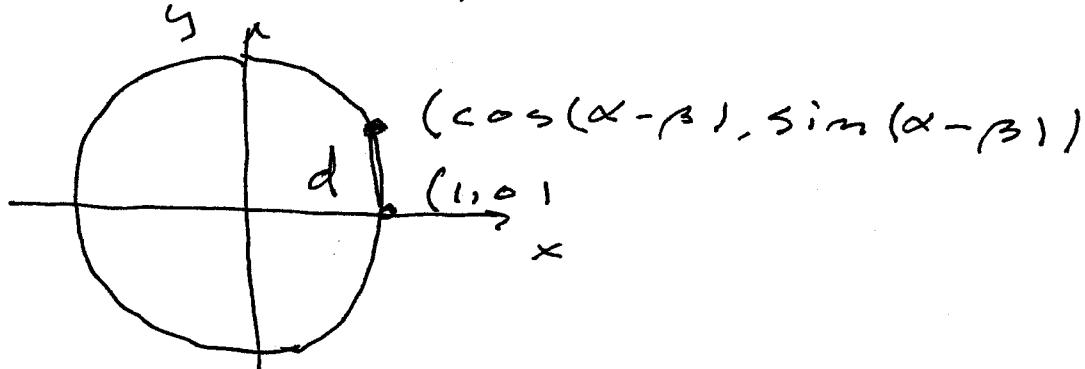
We now prove the trig addition formulas, or equivalently the law of exponents (\*).

□



$$\begin{aligned}
 d^2 &= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 \\
 &= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta \\
 &\quad + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta \\
 &= 2(1 - \cos\alpha\cos\beta - \sin\alpha\sin\beta)
 \end{aligned}$$

Now rotate the points thru the angle  $-\beta$ :



$$\begin{aligned}
 d^2 &= (\cos(\alpha - \beta) - 1)^2 + \sin^2(\alpha - \beta) \\
 &= \cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 \\
 &\quad + \sin^2(\alpha - \beta)
 \end{aligned}$$

(3)

$$d^2 = 2(1 - \cos(\alpha - \beta)).$$

This gives the trig subtraction formula for the cosine:

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

In particular,

$$\cos\left(\alpha - \frac{\pi}{2}\right) = \sin\alpha$$

and

$$\begin{aligned} \sin\left(\alpha - \frac{\pi}{2}\right) &= \cos(\alpha - \pi) \\ &= \cos\alpha \cos\pi + \sin\alpha \sin\pi \\ &= -\cos\alpha. \end{aligned}$$

Replace  $\beta$  by  $-\beta$  in  $(**)$  and use the facts that cos is even and sin is odd ( $\cos(-\alpha) \equiv \cos\alpha$  and  $\sin(-\alpha) \equiv -\sin\alpha$ ):

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta.$$

That's one! Now

$$\begin{aligned} \sin(\alpha + \beta) &= \cos\left(\alpha + \beta - \frac{\pi}{2}\right) \\ &= \cos\left(\left(\alpha - \frac{\pi}{2}\right) + \beta\right) \end{aligned}$$

$$= \cos\left(\alpha - \frac{\pi}{2}\right) \cos \beta -$$

$$- \sin\left(\alpha - \frac{\pi}{2}\right) \sin \beta$$

$$= \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

so we're done ■

Remarks. Since  $\sin \theta = \cos(\theta - \frac{\pi}{2})$

we need only one of the

functions  $\cos$  or  $\sin$ ; I

choose  $\cos$ ! Then  $\tan \theta =$

$$= \sin \theta / \cos \theta = \cos(\theta - \frac{\pi}{2}) / \cos \theta.$$

Can you express  $\cos$  (or  
 $\sin$ ) in terms of  $\tan$ ?

Thus, all the trig functions,  
including  $\cot$ ,  $\sec$ , &  $\csc$ , can  
be expressed in terms of one,  
say  $\cos$ !